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STABILIZATION OF THE POSITION OF A LAGRANGIAN SYSTEM WITH ELASTIC ELEMENTS AND BOUNDED CONTROL, WITH AND WITHOUT MEASUREMENT OF VELOCITIES†

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It is required to position a Lagrangian system whose free and controllable degrees of freedom are elastically linked. The equations of motion of such systems describe, in particular, the dynamics of a robot manipulator with elastic joints. The proposed control laws enable restrictions on the value of the control impulse to be taken into account. In particular, attention is given to the situation in which the velocities are not accessible to measurement. The analysis of the proposed control laws is based on Lyapunov's direct method or, more specifically, on the Barbashin–Krasovskii theorem on asymptotic stability in the large. The proof uses an original method to verify that an auxiliary non-linear function, analogous to the total mechanical energy of a system, closed by a control law, is positive-definite. © 1997 Elsevier Science Ltd. All rights reserved.

1. STATEMENT OF THE PROBLEM

Consider a Lagrangian dynamical system with Lagrangian

$$L = \frac{1}{2}(\dot{q}_1^T D(q_1)\dot{q}_1 + \dot{q}_2^T J\dot{q}_2 + (q_1 - q_2)^T K(q_1 - q_2)) + U(q_1)$$

where q_1 and q_2 are n-dimensional vectors—two groups of generalized coordinates of the system, $D(q_1)$ is a positive-definite $n \times n$ matrix and J and K are constant diagonal matrices with positive diagonal elements. In addition, corresponding to the generalized coordinates q_2 we have a vector u of control forces. Such a system, in particular, simulates the dynamics of an n-link electromechanical manipulator, taking the elasticity of the hinges into account. In that case q_1 is the vector of the angles between the links of the robot, q_2 is the vector of the angles of rotation of the external parts of the electric drive shafts relative to the corresponding supporting links, D is the kinetic energy matrix of the manipulator, J is the kinetic energy matrix of the drive, K is the stiffness matrix of the external parts of the drive shafts, $U(q_1)$ is the potential energy of the manipulator in the gravity field, and u is the vector of control torques applied to the electric drive rotors.

The system may be written as two vector differential equations [1, 2]

$$D(q_1)\ddot{q}_1 + C(q_1, \dot{q}_1)\dot{q}_1 + K(q_1 - q_2) + g(q_1) = 0 \tag{1.1}$$

$$J\ddot{q}_2 + K(q_2 - q_1) = u \tag{1.2}$$

The vector $f(q_1)$ is defined by the torques of the gravity forces and $C(q_1, \dot{q}_1)_{\dot{q}_1}$ is the vector of centrifugal and Coriolis forces. Note that, as is well known (see, for example, [3])

$$\dot{q}_1^T [\dot{D}(q_1) - 2C(q_1, \dot{q}_1)]\dot{q}_1 = 0 \tag{1.3}$$

Let q_{1d} denote the desired (programmed) position of the links.

Given an arbitrary vector $x = [x_1, \ldots, x_n]^T \in R^n$, one can define the norm $\|x\| = \max_i |x_i|$, and then the corresponding norm of the matrix $B = [b_{ij}]_{i,j=1,2,\ldots,n}$ will be $\|B\| = \max_i \sum_j |b_{ij}|$ and $\|K\| = \|\operatorname{diag}\{k_i\}_{i=1}^n\| = \max_i |k_i| = \max_i k_i$.

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The following well-known properties hold: $\| Bx \| \leqslant \| Bx \| \| x \|$; if B^{-1} exists, then $\| Bx \| \geqslant \| B^{-1} \|^{-1} \| x \|$; $\| B_1 + B_2 \| \leqslant \| B_1 \| + \| B_2 \|$; $\| B_2 B_2 \| \leqslant \| B_1 \| \| B_2 \|$; $\| x_1 + x_2 \| \leqslant \| x_1 \| + \| x_2 \|$ for any $x_1, x_2 \in R^n$; B_1, $B_2 \in R^{n \times n}$.

Below we will also use the notation $\lambda_{\max}(B)$ and $\lambda_{\min}(B)$ for the maximum and minimum eigenvalues of a matrix $B \in R^{n \times n}$. Then

$$\| K \| = \lambda_{\max}(K) = \max_i k_i; \quad \| K^{-1} \|^{-1} = \lambda_{\min}(K) = \min_i k_i$$

Let us assume that $a > 0$ and $A > 0$ exist such that

$$\| g(x) - g(y) \| \leqslant \alpha \| x - y \|, \quad \forall x, y \in R^n \tag{1.4}$$

or

$$\| \partial g(q_1) / \partial q_1 \| \leqslant \alpha, \quad \forall q_1 \in R^n; \quad \| g(q_1) \| \leqslant A, \quad \forall q_1 \in R^n \tag{1.5}$$

This assumption has been observed to be true for manipulators with rotational degrees of mobility.

A control law u will be constructed according to the accessible measurement of information. In Section 2 we will assume that both the positions q_2 and their velocities $\dot{q}_2$ are measurable. In Section 3 we will consider a control law based on measurement of the angular positions q_2 alone.

The need for this formulation stems from the fact that it is sometimes very expensive to mount tachometers on a mechanical system. Moreover, as is well known, velocity sensors usually produce less reliable, i.e. more "noisy", information. It is interesting that one can prove asymptotic stability in the large for a system (1.1), (1.2) closed by a control based on angle measurements only, despite the fact that this system makes no allowance for natural friction.

It is important that the proposed control laws involve functions satisfying conditions that enable one to make the control bounded by a prescribed, known quantity. For the practical use of the proposed control laws, it is essential to make allowance for the fact that no amplifier-motor used to control the system can develop a torque exceeding some fixed value (owing to saturation in the output response of the power converter).

In many publications (e.g. [1, 3–5]) bounds on the control are ignored. Bounds have been imposed on only part of the control [6]. A fully bounded control has been assumed, for the case of a Lagrangian system without elastic elements [7]; such a mathematical model admits of full compensation for the gravity force (as indeed was done in [7]), but it is less general than the model (1.1), (1.2) used below (which was proposed in [1, 2]). In a rigid model it is impossible to take into account the elastic vibrations arising in the system, which hinder the exact determination of the coordinates q_1.

Before presenting the main results and actualization of the control laws, we must determine the conditions to be satisfied by the functions $F(x) = [F_1(x_1), \ldots, F_n(x_n)]^T$ used in these laws: $F_i(x_i)$ are continuous, strictly increasing functions, vanishing for a zero, value of the argument such that positive constants β_1, β_2 and γ_2 exist with the following properties

$$\| F(x) \| \geqslant \gamma_1 \| x \|, \quad \text{if} \quad \| x \| \leqslant \beta_1 \tag{1.6}$$

$$\| \partial F(x) / \partial x \| \geqslant \gamma_2, \quad \text{if} \quad \| x \| \leqslant \beta_2 \tag{1.7}$$

$$\| F(x) \| \geqslant \beta_1 \gamma_1, \quad \text{if} \quad \| x \| \geqslant \beta_1 \tag{1.8}$$

We will also assume for simplicity that

$$F(x) = -F(-x) \tag{1.9}$$

Suppose that, in addition $\beta_1 = \beta_2 = \beta > 0$; $F_i(x_i)$ may be chosen either as non-bounded functions, such as

$$F_i(x_i) = \gamma x_i, \quad \gamma_1 = \gamma_2 = \gamma, \quad \beta_1 \text{ and } \beta_2 \text{ are arbitrary numbers} \tag{1.10}$$

or

$$F_i(x_i) = \begin{cases} \gamma x_i, & |x_i| \leq \beta \\ \gamma\beta + \varepsilon(x_i - \beta), & x_i > \beta \\ -\gamma\beta + \varepsilon(x_i + \beta), & x_i < -\beta \end{cases} \tag{1.11}$$

$$\beta_1 = \beta_2 = \beta, \quad \gamma_1 = \gamma_2 = \gamma, \quad \varepsilon > 0$$

or as bounded ones, such as

$$F_i(x_i) = \frac{\beta\gamma}{\operatorname{arctg}\beta} \operatorname{arctg} x \tag{1.12}$$

$$\beta_1 = \beta_2 = \beta, \quad \gamma_1 = \gamma, \quad \gamma_2 = \frac{\beta\gamma}{(1+\beta^2)\operatorname{arctg}\beta}$$

The first example (1.10) is the ideal response curve of an amplifier, while the others ((1.11) and (1.12)) are less idealized models. The most interesting example is (1.12), since in that case

$$|F_i(x_i)| \leq \frac{\beta\gamma}{\operatorname{arctg}\beta} \frac{\pi}{2} = \text{const.}$$

We can now proceed to define the control laws and analyse them in the stability-theoretic sense.

2. ASYMPTOTIC STABILIZATION OF A SYSTEM WITH GENERALIZED VELOCITIES $\dot{q}^2$ ACCESSIBLE TO MEASUREMENT

We propose the following control law

$$u = F(-(q_2 - q_{2d}) - K_v \dot{q}_2) + g_d \quad (g_d = g(q_{1d})) \tag{2.1}$$

where $F(x)$ is the vector function defined at the end of Section 1, satisfying conditions (1.6)–(1.9) with certain positive constants γ_1, γ_2 and β, the choice of which will be explained below and, in physical terms, will mean that the control torques prevail over the torques due to gravity, K_v is some positive-definite diagonal matrix and q_{2d} is a vector defining the fixed desired position of the coordinates q_2, which is calculated from the desired position of the coordinates q_1 as follows:

$$q_{2d} = q_{1d} + K^{-1} g_d \tag{2.2}$$

Thus, of the whole dynamical model, the known elements are the stiffnesses and a vector defining the torques of the gravity forces in the desired position. Only $2n$ of the $4n$ phase coordinates of system (1.1), (1.2) are assumed to be accessible to measurement.

We introduce the following notation

$$\begin{aligned} &q_{12} = -q_{21} = q_1 - q_2, \quad q_{12d} = -q_{21d} = q_{1d} - q_{2d} \\ &x = q_1 - q_{1d}, \quad y = q_2 - q_{2d} \\ &\Delta g(x) = g(x + q_{1d}) - g_d \end{aligned} \tag{2.3}$$

Proposition 2.1. If the coefficients β_1 and γ_1 in (1.6) and (1.8) are chosen so that (1.4) and (1.5) are satisfied, and in addition

$$\gamma_1 > \alpha(1 - \alpha\|K^{-1}\|)^{-1} > 0, \quad \gamma_1\beta_1 > A + \|g_d\| \tag{2.4}$$

then the closed system (1.1), (1.2), (2.1) has a unique equilibrium position, which is precisely the desired position: $q_1 = q_{1d}$, $q_2 = q_{2d}$ of (2.2).

Proof. An equilibrium position is defined by the system

$$g(q_1)=Kq_{21},\quad Kq_{21}=F(-y)+g_d \tag{2.5}$$

In view of (2.2), system (2.5) may be rewritten as

$$\Delta g(x)=K(y-x),\quad K(y-x)=F(-y) \tag{2.6}$$

Equations (2.5) are satisfied if and only if

$$y=X,\quad \Delta g(x)=F(-X)\quad (X=x+K^{-1}\Delta g(x)) \tag{2.7}$$

The second relationship in (2.7) is valid for some x only if

$$\|\Delta g(x)\|=\|F(-X)\| \tag{2.8}$$

If $\|X\|\leqslant\beta_1$, then, by (1.6) and (1.4)

$$\|F(-X)\|\geqslant\gamma_1\|X\|\geqslant\gamma_1(\|x\|-\|K^{-1}\|\,\|\Delta g(x)\|)\geqslant\gamma_1(1-\alpha\|K^{-1}\|)\|x\|$$

On the other hand, by (1.4), $\|\Delta g(x)\|\leqslant\alpha\|x\|$. Then Eq. (2.8) will hold only if $\alpha\|x\|\geqslant\gamma_1(1-\alpha\|K^{-1}\|)\|x\|$, but in view of the first inequality in (2.4) this is possible only if $\|x\|=0$.

If $\|X\|\geqslant\beta_1$, then, by (1.8), $\|F(-X)\|\geqslant\beta_1\gamma_1$, and it follows from (1.5) that $\|\Delta g(x)\|\leqslant\|g_d\|+A$. Then, for condition (2.8) to hold, necessarily $\|g_d\|+A\geqslant\beta_1\gamma_1$—but this is impossible because of the second relationship in (2.4).

Thus, condition (2.7) will hold only if $\|x\|=0$, that is, $x=0$. It then follows from the first equality in (2.7) that $y=0$. This means that Eqs (2.6) will hold only if $x=0, y=0$.

Note that the torques of the gravity forces may be expressed in terms of the potential energy $U(q_1)$ by the formula

$$g(q_1)=[g_1(q_1),\ldots,g_n(q_1)]^T=\partial U/\partial q_1=[\partial U/\partial q_{1_1},\ldots,\partial U q_{1_n}]^T$$

By definition

$$\int_0^y F^T(\eta)d\eta=\sum_{i=1}^n\int_0^{y_i}F_i(\eta_i)d\eta_i$$

Consider the function

$$P(q_1,q_2)=\frac{1}{2}q_{12}^TKq_{12}-\frac{1}{2}q_{12d}^TKq_{12d}+U(q_1)-U_d+(q_{2d}-q_2)^Tg_d+\int_0^{q_{2d}-q_2}F^T(\eta)d\eta \tag{2.9}$$

where we have introduced the notation $U_d=U(q_{1d})$.

Apart from the last term, which is introduced by analogy with previously chosen terms [7], $P(q_1,q_2)$ consists of the potentials of the forces acting in the closed system (1.1), (1.2), (2.1). Taking (2.2) and (2.9) into account, we can reduce (2.2) and (2.9) to the form

$$P=\frac{1}{2}(x-y)^TK(x-y)+U(x+q_d)-U_d+x^Tg_d+\int_0^{-y}F^T(\eta)d\eta$$

If $F(\eta)$ is chosen to be the vector of functions (1.10), the reduced function P is the same as that considered in [4].

Proposition 2.2. If inequalities (2.4) hold, the function $P(q_1,q_2)$ has the following properties

1. $P(q_1,q_2)$ has a unique stationary point S: $q_1=q_{1d}$, $q_2=q_{2d}$.
2. If

$$\lambda_{\min}(K)>\delta\alpha\ \text{ and }\ \gamma_2>\delta\alpha\quad\left(\delta=\lambda_{\max}\left(\begin{Vmatrix}1&1\\1&2\end{Vmatrix}\right)=\frac{3+\sqrt{5}}{2}\right) \tag{2.10}$$

where α is from (1.4) and γ_2 from (1.7), then $P(q_1,q_2)$ is convex in the β_2-neighbourhood of S, so that S is a local minimum point.

3. $P(q_1, q_2) \to +\infty$ as $\| [q^T{}_1, 1^T{}_2]^T \| \to +\infty$.
4. $P(q_1, q_2) > P(q_{1d}, q_{2d}) = 0$ for any $(q_1 q_2) \neq (q_{1d}, q_{2d})$, that is, S is a global minimum point.

Proof. 1. The stationarity condition $\partial P/\partial q_1 = 0$, $\partial P/\partial q_2 = 0$ leads directly to system (2.5), and one can use the result of Proposition 2.1.

2. We have to consider the neighbourhood $\|x\| \leq \beta_2$, $\|y\|\,\beta_2$. In this neighbourhood the matrix of second derivatives of $P(q_1, q_2)$ has a lower bound in the sense of quadratic forms

$$\frac{\partial^2 P(q_1,q_2)}{(\partial[q_1^T,q_2^T]^T)^2} \geq \left\| \begin{matrix} K+\partial g/\partial q_1 & -K \\ -K & K+\gamma_2 E \end{matrix} \right\|$$

Now apply the proposition of [5].

3. Make the non-singular change of variables $\xi = x - y$, $\eta = x + y$ and use (2.3). Then

$$P = \frac{1}{2}\xi^T K\xi + \int_0^{(\xi-\eta)/2} F^T(\zeta)d\zeta + U\left(\frac{\xi+\eta}{2}+q_{1d}\right) - U_d - \left(\frac{\xi+\eta}{2}\right)^T g_d$$

The direction $[a^T, b^T]^T$ in (ξ, η) spaces may be chosen so that

$$\|a\| = \beta,\ \|b\| \leq \beta \ \text{ or } \ \|a\| \leq \beta,\ \|b\| = \beta \tag{2.11}$$

When all such values of a and b are substituted, the end of the vector $[a^T, b^T]^T$ describes the boundary of the domain considered in property 2.

Let us consider the behaviour of the function P on the ray $[\xi^T, \eta^T]^T = t[a^T, b^T]^T$ for $t \in [1, \infty)$. We obtain

$$\frac{dP}{dt} = t(a^T Ka) + \left(\frac{a-b}{2}\right)^T F\left(\frac{a-b}{2}t\right) + \left(\frac{a+b}{2}\right)^T \Delta g\left(\frac{a+b}{2}t\right)$$

If $\| a \| > 0$ it follows from (1.8), (1.9) and (1.5) that $dP/dt \simeq t(a^T Ka)$ as $t \to \infty$, that is, P increases at a rate proportional to t, beginning from some value of t which increases at $\| a \|$ approaches zero. If $\| a \| = 0$, then $\| b \| = \beta$ by (2.11), and so (using (1.9))

$$\frac{dP}{dt} = -\frac{b^T}{2} F\left(-\frac{b}{2}t\right) + \frac{b^T}{2}\Delta g\left(\frac{b}{2}t\right) \geq \frac{\beta}{2}(\gamma_1\beta - [A + \|g_d\|]) = \text{const} > 0$$

Thus, P increases at least at a constant rate proportional to $(\gamma_1\beta - [A + \| g_d \|])$ along any direction in (ξ, η) space, hence also in (x, y) space, beginning from some fixed time t^*. This completes the proof of property 3.

4. Consider the neighbourhood of the origin in (ξ, η) space bounded by the surface $[\xi^T, \eta^T]^T = [a^T, b^T]^T t^{**}$, where a and b run through all possible values in (2.11) and $t^{**} > t^*$, t^* being the same is in the proof of 3. The function $P(q_1, q_2)$ is continuous, as is clear from (2.9); consequently, P takes its minimum value in this compact neighbourhood of the origin either at the local minimum point $[x^T, y^T]^T = [0^T, 0^T]^T$, where $P = 0$, or on the surface. The possibility that p might take a negative value at some point on the boundary of the neighbourhood is refuted with the help of property 3, which implies that a finite increase in t^{**} may increase the value of P on the surface (which itself is modified by this increase) by any finite number in the positive sense. This completes the proof of property 4.

Theorem 2.1. If conditions (1.4)–(1.9), (2.4), (2.10) are satisfied, the closed system (1.1), (1.2), (2.1) has a unique equilibrium position, which coincides with the desired position: $q_1 = q_{1d}$, $q_2 = q_{2d}$, where q_{2d} is as defined in (2.2); moreover, the equilibrium position is asymptotically stable in the large.

Proof. Consider the following auxiliary function

$$V(q_1, q_2, \dot{q}_1, \dot{q}_2) = \tfrac{1}{2}(\dot{q}_1^T D(q_1)\dot{q}_1 + \dot{q}_2^T J\dot{q}_2) + P(q_1, q_2)$$

bearing (2.9) in mind.

By Proposition 2.2, this is a Lyapunov function for the closed system. It is the sum of the kinetic energy and an analog of the potential energy of the system.

The rate of change of $V(q_1, q_2, \dot{q}_1, \dot{q}_2)$ along a trajectory of the closed system is

$$\dot{V}(q_1, q_2, \dot{q}_1, \dot{q}_2) = \tfrac{1}{2}\dot{q}_1^T \dot{D}(q_1)\dot{q}_1 + \dot{q}_1^T D(q_1)\ddot{q}_1 + \dot{q}_2^T J\ddot{q}_2 + q_{12}^T K\dot{q}_{12} +$$

$$+\dot{q}_1^T g(q_1) - \dot{q}_2^T g_d + F^T(-y)(-\dot{q}_2) = \tfrac{1}{2}\dot{q}_1^T \dot{D}(q_1)\dot{q}_1 + \dot{q}_1^T[-C(q_1,\dot{q}_1)\dot{q}_1] +$$

$$+\dot{q}_1^T[-g(q_1) + Kq_{21}] + \dot{q}_2^T[-Kq_{21} + F(-y - K_v\dot{q}_2) + g_d] + q_{12}^T K\dot{q}_{12} + \dot{q}_1^T g(q_1) -$$

$$-\dot{q}_2^T g_d - \dot{q}_2^T F(-y) = -\dot{q}_2^T[F(y + K_v\dot{q}_2) - F(y)]$$

The cancellations follow from (1.3) and (1.9).

By the fact that $F_i(y_i)$ is strictly increasing, we have $\dot{V} \leqslant 0$, with $\dot{V} = 0$ only if $\dot{q}_2 = 0$, i.e. $q_2 = \dot{q}_{2c} =$ const. It then follows from (2.1) that $u = u_c =$ const and from (1.2) that $q_1 = q_{1c} =$ const. Then, by Proposition 2.1, $q_1 = q_{1d}$, $q_2 = q_{2d}$, and the set $\dot{V} = 0$ does not contain other complete trajectories. To complete the proof of the theorem, we need only apply the Barbashin–Krasovskii theorem on asymptotic stability in the large.

Corollary 2.1. If there are no gravity forces in the system, Eqs (1.1), (1.2), (2.1) are asymptotically stable in the large for all positive-definite diagonal matrices K_v and all strictly increasing, continuous functions $F(x)$ that vanish for zero values of the argument.

The class of equations considered up to this point has a drawback, namely the requirement that $F_i(y_i)$ be strictly increasing functions. It should be obvious from the foregoing arguments that this condition is necessary only to prove that there are no complete trajectories (other than the equilibrium position $q_1 = q_{1d}$, $q_2 = q_{2d}$) in the set $\dot{V} = 0$. One can eliminate this drawback, that is, permit functions $F_i(y_i)$ such that $F_i(y_i) =$ const $\geqslant \beta_1\gamma_1$ for $|y_i| \geqslant \beta_3 \geqslant \beta_1$, by replacing (2.1) with an equation

$$u = -F(y) - F^*(\dot{q}_2) + g_d \tag{2.12}$$

where $F_i(y_i)$ may be the same as in [6] (of course, satisfying conditions (1.6)–(1.9)) or, as in (1.11) with $\varepsilon = 0$. $F_i^*(y_i)$ are arbitrary continuous odd functions whose derivative vanishes at zero.

Theorem 2.2 If conditions (1.4)–(1.9), (2.4) and (2.10) are satisfied, the closed system (1.1), (1.2), (2.12) has a unique equilibrium position, which coincides with the desired position: $q_1 = q_{1d}$, $q_2 = q_{2d}$; moreover, the equilibrium position is asymptotically stable in the large.

Proof. The function $P(q_1, q_2)$ for the new system is not just an analogue of the potential energy but a *bona fide* potential energy, that is, its derivatives with respect to the coordinates define the forces acting in the system. The proof of Theorem 2.2 differs from that of Theorem 2.1 only at the final stage. The rate of change of the function $V(q_1, q_2, \dot{q}_1, \dot{q}_2)$ along a trajectory of the new system takes the form

$$\dot{V} = -\dot{q}_2^T[\{F(y) + F^*(\dot{q}_2)\} - F(y)] = -\dot{q}_2^T F^*(\dot{q}_2)$$

And now the statement that $\dot{V} = 0$ only when $\dot{q}_2 = 0$ remains valid regardless of the fact that the coordinates F^* may remain constant over finite intervals (of course, outside an arbitrarily small neighbourhood of zero). The rest of the proof is completed as before.

3. ASYMPTOTIC STABILIZATION WHEN THE GENERALIZED VELOCITIES ARE NOT ACCESSIBLE TO MEASUREMENT

We will consider the following control law

$$u = F(q_3 - q_2) + g_d \tag{3.1}$$

The components of the vector F may again be taken in the form (1.10), (1.11) or (1.12), that is, they increase continuously, so that conditions (1.6)–(1.9) are satisfied.

The law (3.1) does not involve the velocities, but on the other hand it depends essentially on the vector q_3, which is evaluated in parallel with the motion and the measurement of q_2 by numerical or mechanical solution of the equation

$$\dot{q}_3 = -G^{-1}(F(q_3 - q_2) + \varkappa(q_3 - q_{2d})) \tag{3.2}$$

where G and $\varkappa$ are positive-definite diagonal matrices and q_{2d} is as in (2.2).

The scheme and methods for investigating the law (3.1) basically repeat the investigation of the law (2.1) in the previous section. One first proves analogues of Propositions 2.1 and 2.2, and then arrives at the main theorem, which is analogous to Theorem 2.1.

Proposition 3.1. If the coefficients β_1, γ_1 and α are chosen so that

$$\gamma_1 > \alpha(1-\alpha\|K^{-1}+\varkappa^{-1}\|)^{-1} > 0, \quad \gamma_1\beta_1 > A+\|g_d\| \tag{3.3}$$

then system (1.1), (1.2), (3.1), (3.2) has a unique equilibrium position and it coincides with the desired position: $q_1 = q_{1d}$, $q_2 = q_{2d}$ as in (2.2), $q_3 = q_{2d}$.

The proof is analogous to that of Proposition 2.1. After appropriate reduction one obtains, instead of (2.7)

$$y = X, \quad z = Z, \quad F(Z-X) = \Delta g(x) \tag{3.4}$$

where we have introduced the notation $z = q_3 - q_{2d}$, $Z = -\varkappa^{-1}\Delta g(x)$. Now, estimating the right- and left-hand sides of the third equation in (3.4) in norm, one reaches the conclusion $x = 0, y = 0, z = 0$.

By analogy with (2.9), we introduce the function

$$P(q_1,q_2,q_3) = \frac{1}{2}q_{12}^TKq_{12} - \frac{1}{2}q_{12d}^TKq_{12d} + U(q_1) - U_d +$$

$$+(q_{2d}-q_2)^Tg_d + \int_0^{q_3-q_2} F^T(\eta)d\eta + \frac{1}{2}(q_3-q_{2d})^T\varkappa(q_3-q_{2d}) \tag{3.5}$$

Proposition 3.2. If inequality (3.3) holds, the function $P(q_1, q_2, q_3)$ has the following properties.

1. $P(q_1, q_2, q_3)$ has a unique stationary point S: $q_1 = q_{1d}$, $q_2 = q_{2d}$, $q_3 = q_{2d}$.
2. If

$$\lambda_{\min}(K) > \delta\alpha, \quad \gamma_2 > \delta\alpha, \quad \lambda_{\min}(\varkappa) > \delta\alpha \quad \left(\delta = \lambda_{\max}\left(\begin{Vmatrix}1 & 1 & 1\\ 1 & 2 & 2\\ 1 & 2 & 3\end{Vmatrix}\right)\right) \tag{3.6}$$

with α as in (1.4) and γ_2 as in (1.7), then a finite neighbourhood of the point S exists in which $P(q_1, q_2, q_3)$ is convex, and therefore S is a local minimum point.

3. $P(q_1, q_2, q_3) \to \infty$ as $\|[q_1^T, q_2^T, q_3^T]^T\| \to +\infty$.
4. $P(q_1, q_2, q_3) > P(q_{1d}, q_{2d}, q_{3d}) = 0$ everywhere except at the point S.

The proof is analogous to that of Proposition 2.2. The main difference is as follows. In proving property 2 one considers the neighbourhood $\|z\| \leqslant \beta/3$, $\|x+y\| \leqslant 2\beta/3$, $\|x-y\| \leqslant 2\beta/3$. In this neighbourhood $\|q_3 - q_2\| \leqslant \beta$, and so, using (1.7), one can estimate the matrix of second derivatives of $P(q_1, q_2, q_3)$ (in the sense of quadratic forms) as follows:

$$\frac{\partial^2 P(q_1,q_2,q_3)}{(\partial[q_1^T,q_2^T,q_3^T]^T)^2} \geqslant \begin{Vmatrix} K+\partial g/\partial q_1 & -K & 0\\ -K & K+\gamma_2E & -\gamma_2E\\ 0 & -\gamma_2E & \varkappa+\gamma_2E\end{Vmatrix}$$

In the proof of property 3, the behaviour of P is considered on rays $[x^T-y^T, x^T+y^T, z^T]^T = [a^T, b^T, c^T]^Tt$, $t\in[1,\infty)$, $\|c\| \leqslant \beta/3$, $\|a\| \leqslant 2\beta/3$, $\|b\| \leqslant 2\beta/3$, at least one of these inequalities being strict.

Theorem 3.1. If conditions (1.4)–(1.9), (3.3), (3.6) are satisfied, the closed system (1.1), (1.2), (3.1), (3.2) has a unique equilibrium position, which coincides with the desired position: $q_1 = q_{1d}$, $q_2 = q_{2d}$, $q_3 = q_{2d}$, where q_{2d} is as defined in (2.2); moreover, this position is asymptotically stable in the large.

Proof. We will consider the following auxiliary function

$$V(q_1,q_2,q_3,\dot q_1\dot q_2) = \tfrac{1}{2}(\dot q_1^TD(q_1)q_1 + \dot q_2^TJ\dot q_2) + P(q_1,q_2,q_3)$$

and use expression (3.5).

By Proposition 3.2, $V(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2)$ is a Lyapunov function for the system.

The rate of change of $V(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2)$ along trajectories of the closed system (1.1), (1.2), (3.1), (3.2) is

$$\dot{V}(q_1,q_2,q_3,\dot{q}_1,\dot{q}_2)=\tfrac{1}{2}\dot{q}_1^T\dot{D}(q_1)\dot{q}_1-\dot{q}_1^TC(q_1,\dot{q}_1)\dot{q}_1+\dot{q}_1^T[-g(q_1)+Kq_{21}]+$$
$$+\dot{q}_2^T[-Kq_{21}+F(q_3-q_2)+g_d]+(\dot{q}_1-\dot{q}_2)^TKq_{12}+(q_3-q_{2d})^T\varkappa\dot{q}_3+$$
$$+\dot{q}_1^Tg(q_1)-\dot{q}_2^Tg_d+\dot{q}_3^TF(q_3-q_2)-\dot{q}_2^TF(q_3-q_2)=-\dot{q}_3^TG\dot{q}_3$$

The arguments now take relations (1.1), (1.2), (1.3), (3.1) and (3.2) into consideration. It is obvious that, since G is positive-definite, we have $\dot{V}\leqslant 0$, and if $\dot{V}=0$, then $\dot{q}_3=0$, that is, $q_3=q_{3c}=$ const. Then, by (3.2), $F(q_3-q_2)=-\varkappa(q_{3c}-q_{2d})=$ const. Since the components of the vector $F(x)$ are strictly monotone functions, this implies that $q_2=q_{2c}=$ const, and it then follows from (1.2) that $q_1=q_{1c}$ and, by Proposition 3.1, $q_1=q_{1d}$, $q_2=q_{2d}$, $q_3=q_{2d}$ and the set $\dot{V}=0$ does not contain other complete trajectories of the system. To complete the proof it remains to apply the Barbashin–Krasovskii theorem on asymptotic stability in the large.

If the torques of the gravity forces are negligible, the stability conditions may be weakened.

Corollary 3.1. If there are no gravity forces in the system, then Eqs (1.1), (1.2), (3.1), (3.2) are asymptotically stable in the large for any positive-definite diagonal matrices G and κ and any monotone increasing continuous functions $F(x)$ that vanish at zero.

4. REMARKS

1. The model (1.1), (1.2) of [1, 2] may be replaced by the more general model of [4]

$$D(q_1)\ddot{q}_1+B(q_1)\ddot{q}_2+C_1(q_1,\dot{q}_1,\dot{q}_2)\dot{q}_1+C_2(q_1,\dot{q}_1)\dot{q}_2+K(q_1-q_2)+g(q_1)=0$$
$$J\ddot{q}_2+B^T(q_1)\ddot{q}_1+C_3(q_1,\dot{q}_1)\dot{q}_1+K(q_2-q_1)=u$$

The same Theorems 2.1, 2.2 and 3.1 may be proved for this model too. The proof is slightly more complicated: the terms $(\dot{q}_1^TD(q_1)\dot{q}_1+\dot{q}_2^TJ\dot{q}_2)/2$ in the auxiliary functions V must be replaced by a new kinetic energy

$$\frac{1}{2}[\dot{q}_1^T\dot{q}_2^T]\begin{Vmatrix}D(q_1) & B(q_1)\\ B^T(q_1) & J\end{Vmatrix}\begin{Vmatrix}\dot{q}_1\\ \dot{q}_2\end{Vmatrix}$$

2. If we compare the control law of [4]

$$u=-K_p(q_2-q_{2d})-K_v\dot{q}_2+g_d$$

with the control laws (2.1), and especially (2.12), it can be seen that the laws proposed in this paper are extensions of the former, providing a closer approximation to the actual responses of an amplifier-motor system.

3. A similar remark holds if one compares the control law of [5] with (3.1). But even more: the auxiliary equation (3.2) proposed here (which is an estimator of the generalized velocities) is of order half as much as in [5] and is most probably easier to implement. There have been earlier studies [8–10] of the asymptotic stabilization of non-linear mechanical systems without measurement of the generalized velocities. In these studies, the number of auxiliary differential equations that must be solved during the control process was equal to the number of controls. In a study of asymptotic stabilization of a Lagrangian system by bounded controls with velocity measurement [11], the number of control signals was equal to the number of degrees of freedom , but there were not external potential forces.

4. The energy method is widely used to investigate the stability of non-linear mechanical systems with dissipation. We may mention studies of the asymptotic stabilization of a Lagrangian system which is quite general relative to velocities [12], and also relative to position coordinates and velocities [13]. Mention should also be made of [14, 15].

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